Coulomb Scattering of Dirac Particles*

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The expressions for the Coulomb scattering amplitudes F and G up to α^4 terms ($\alpha=Z/137$) for arbitrary $q=\alpha/\beta$ are given. These expressions are used to evaluate the cross section $d\sigma/d\Omega$ and the asymmetry function S at small angles. The behavior of F and G as well as $d\sigma/d\Omega$ and S near $\theta=180°$ is also discussed. The approximation is then extended to a screened potential by introducing a cutoff parameter $e^{-\mu n}$ ($\mu<<1$) in the partial-wave series. The scattering amplitudes and $d\sigma/d\Omega$ and S are compared with the corresponding Coulomb case.

I. INTRODUCTION

In studying the effect of finite scatterer thickness on the measurement of electron polarization by means of Mott scattering,[1] we have been led to investigate the analytic behavior of the Mott scattering amplitudes f and g (or F and G) at small angles. In the actual case, of course, the Coulomb amplitudes are modified at small angles because of screening by the atomic electrons. This effect has been calculated numerically first by Mohr and Tassie,[2] and more recently by Lin, Sherman, and Percus[3] from solutions of the Dirac equation in Hartree-like potentials. In order to compare with the unscreened Mott amplitudes, we have developed methods for obtaining these Mott amplitudes accurately at small angles, and the results are given in a separate paper.[4] These expressions for F and G are valid for all values of $x=\sin\theta/2$ except at $x=1$.

In this paper we evaluate the cross section $d\sigma/d\Omega$ and the asymmetry function S using the results of paper I. A comparison between $d\sigma/d\Omega$ and S obtained by exact numerical calculation and by our approximation is given. We also study the behavior of F and G as well as $d\sigma/d\Omega$ and S near $\theta=180°$. Finally, we extend our small-angle approximation to a screened potential.

II. SMALL-ANGLE APPROXIMATION

According to the results of Sec. 3 of Paper I, the ratios F_1/F_0 and G_1/G_0 up to α^4x^2 and α^4x^{2-2iq} are

$$F_1/F_0=\frac{i\pi\alpha^2x}{1+2iq}e^{i\psi}-\frac{i\alpha^2x^2}{2q(1+iq)}\left[\frac{\pi^2\alpha^2}{4}-(1+2iq)\right]+\frac{i\alpha^2\pi^2x^{2-2iq}}{2q}\frac{e^{\pi q}}{\sinh\pi q}\left\{\frac{\alpha^2}{2i\pi}\Omega-\frac{q}{\pi(1-2iq)}\right\}\tag{2.1}$$

and

$$\frac{G_1}{G_0}=\frac{\pi x\alpha^2}{2q}e^{i\psi}-\frac{x^2\alpha^2}{2q^2}\left[\frac{\pi^2\alpha^2}{4}-(1+2iq)\right]+\frac{\alpha^2\pi^2}{2q^2}x^{2-2iq}\frac{e^{\pi q}}{\sinh\pi q}\left[\frac{\alpha^2}{2i\pi}\Omega-\frac{q}{\pi(1-2iq)}\right],\tag{2.2}$$

where

$$\Omega=\frac{\Gamma'(1+2iq)}{\Gamma(1+2iq)}-\frac{\Gamma'(1-iq)}{\Gamma(1-iq)},$$

$$\psi=2\arg\Gamma(1+iq)-2\arg\Gamma(\tfrac{1}{2}-iq).$$

Here $\alpha=Z/137$ and $q=\alpha/\beta$.

It is interesting to notice from (2.1) and (2.2) that the behavior of F_1/F_0 and G_1/G_0 for large $|q|$ (nonrelativistic limit) is markedly different for positrons ($q<0$) and electrons ($q>0$). For positrons, the oscillatory terms vanish in the limit $|q|\to\infty$, whereas for electrons they do not. This behavior has been shown to be true only up to α^4, but can be shown to be true for all powers of α^2. It leads to different behavior of $d\sigma/d\Omega$ and S for electrons and positrons in the nonrelativistic limit as noticed by Fradkin, Weber, and Hammer[5] for the Dirac case and by Rawitscher[6] for the corresponding Klein-Gordon case.

We can now write down the cross section and the asymmetry function for the small-angle region. The cross section is

$$\frac{d\sigma(\theta)}{d\Omega}\simeq\frac{q^2}{4k^2x^4}\left\{1+\pi x\frac{\alpha^2}{q}\cos\psi+\frac{\alpha^2}{q}x^2\cos\eta\times\left(\frac{\pi^2\alpha^2}{4q}a_0-\frac{c}{1+4q^2}\right)-\frac{2\alpha^2}{1+4q^2}x^2c'\sin\eta\right\},\tag{2.3}$$

where

$$\eta=2q\ln x,$$

$$c=\pi e^{\pi q}/\sinh\pi q,$$

$$c'=c\{1-[(1+4\alpha^2)/4]\operatorname{Re}\Omega\},$$

$$a_0=\frac{2c}{\pi^2}\operatorname{Im}\Omega=\frac{2c}{\pi^2}\left\{-\frac{3}{4q}+\frac{\pi}{4\tanh\pi q}(3+\tanh^2\pi q)\right\}.$$

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[1] D. Lazarus and J. S. Greenberg (private communication).

[2] C. B. O. Mohr and L. J. Tassie, Proc. Phys. Soc. (London) **A67**, 711 (1954).

[3] S. R. Lin, N. Sherman, and J. K. Percus, Nucl. Phys. **45**, 492 (1963). See also S. R. Lin, Phys. Rev. **133**, A965 (1964).

[4] R. L. Gluckstern and S. R. Lin, J. Math. Phys. (to be published). Hereafter this paper will be referred to as Paper I and formulas in this paper will be quoted with the prefix I.

[5] D. M. Fradkin, T. A. Weber, and C. L. Hammer, Ann. Phys. (N. Y.) **27**, 338 (1964).

[6] G. Rawitscher, Phys. Letters **9**, 337 (1964).

TABLE I. Ratio of cross section in small-angle approximation to that of exact numerical calculation.

Z	13	79	80
v/c \ θ	0.4	0.6889	0.4
5°	1.0000	1.0002	1.0002
10°	1.0000	1.0010	0.9998
15°	1.0000	1.0009	0.9997

Our results agree with those of Drell and Pratt[7] which are given only for the cross section in the limit $\beta=1$.

In Table I, we compare results obtained with the approximation (2.3) and exact numerical calculation performed on an IBM-709 computer for typical values of Z and β. As a general result, the error is less than of order 0.1%. This is remarkable considering the fact that for $Z=80$, $\alpha=0.58$.

The asymmetry function for the small-angle region is similarly obtained:

$$S\simeq 2x^2(1-\beta^2)^{1/2}\left\{\frac{\pi\alpha^2(\sin\psi-2q\cos\psi)}{2q(1+4q^2)}+x\alpha^2\left[\frac{1-\frac{3}{4}\pi^2\alpha^2/(1+4q^2)}{2q(1+q^2)}-\frac{\pi^2\alpha^2\cos\psi}{2q^2(1+4q^2)}(\sin\psi-2q\cos\psi)\right]+\frac{\alpha^2}{2}x\left[\left(\frac{-2c'+c}{1+4q^2}-\frac{\pi^2\alpha^2}{4q}a_0\right)\cos\eta+\left(\frac{2q^2c'+c}{q(1+4q^2)}-\frac{\pi^2\alpha^2}{4q^2}a_0\right)\sin\eta\right]\right\}. \quad (2.4)$$

In Table II, we compare our results with exact numerical calculations. Here the agreement is not expected to be as good as for the cross section, since the main Born term leads to no asymmetry. Nevertheless, the small-angle behavior is of the form

$$S/2x^2=J_0+x(J_1+A_1\cos\eta+B_1\sin\eta)\,, \quad (2.5)$$

where J_0 and J_1 are given correctly for all α and q in (2.4), but A_1 and B_1 are available only to order α^4. For those values of α and β for which A_1 and B_1 are not well convergent, two exact values of S at different angles can be used to determine A_1 and B_1. Then accurate values of S can be obtained at small angles by using (2.5) with these values of A_1 and B_1.

III. BEHAVIOR OF $d\sigma/d\Omega$ AND S NEAR 180°

In this section, we shall develop approximations (up to α^2) for $x\simeq 1$. Here we will show that for high-energy electrons ($T>1$ MeV) S has its maximum near 180° and the angle at which this maximum occurs approaches 180° as the energy increases. This behavior of S has been recognized by many people[8]; we shall derive its approximate analytic form.

Our starting point is I_1 in (I-3.8):

$$I_1=\frac{i\alpha^2}{4}\frac{D_\epsilon}{\Gamma(1+2iq)}\int_0^1(1-t)^{2iq+1}t^{\epsilon-iq-1}\times\left(\frac{1}{[(1+t)^2-4ty^2]^{1/2}}-\frac{1}{1-t}\right)dt\,, \quad (3.1)$$

where $D_\epsilon\equiv(-i\pi+\partial/\partial t)_{\epsilon=0}$, and where $y=\cos\theta/2$ vanishes at $\theta=\pi$. We can expand (3.1) directly in powers of y^2 and express each term as a well convergent hypergeometric function of argument $\frac{1}{2}$. Since $d\sigma/d\Omega$ and S are desired only to lowest order in y^2, we need only the first term in I_1:

$$I_1{}^0=\frac{\alpha^2}{q}F_0\left\{\pi\left[\frac{e^{\pi q}}{2\sinh\pi q}+\frac{\pi^{1/2}}{2}ae^{i(\psi/2)}\left(\frac{e^{\pi q}}{\sinh\pi q}-\frac{i(1-iq)}{\pi(1+q^2)}\right)-\frac{1+2iq}{4i(1+iq)}\frac{\partial}{\partial c}F(2iq+2,1,c;\tfrac{1}{2})\Big|_{c=2+iq}\right]\right\}, \quad (3.2)$$

where $a=(q\coth\pi q)^{1/2}$. To the same approximation, one can write down a similar expression for G_1 by using (I-2.14) and get

$$J_1{}^1=-\frac{i\alpha^2}{q}G_0\left[\frac{\pi^{1/2}}{4}ae^{i\psi/2}\left(\frac{i\pi e^{\pi q}}{\sinh\pi q}-\frac{i}{q}+\frac{1-iq}{1+q^2}+\frac{2-iq}{4+q2}\right)-\frac{1}{8}\frac{1+2iq}{(2+iq)}\frac{1}{(1+iq)}\times\frac{\partial}{\partial c}F(2iq+2,3,c;\tfrac{1}{2})\Big|_{c=iq+3}\right]. \quad (3.3)$$

To order α^2, F and G are therefore of the form

$$\begin{aligned}F&=F_0\left\{1+\frac{\alpha^2}{q}\frac{I_1{}^0}{F_0}\right\}=F_0\left\{1+\frac{\alpha^2}{q}(A+iB)\right\},\\ G&=G_0\left\{1+\frac{\alpha^2}{q}\frac{J_1{}^1}{G_0}\right\}=G_0\left\{1+\frac{\alpha^2}{q}(E+iH)\right\},\end{aligned} \quad (3.4)$$

where A, B, E, H are complicated functions of q obtained from (3.2) and (3.3). The cross section is

$$\frac{d\sigma}{d\Omega}\simeq\frac{q^2(1-\beta^2)}{4k^2}\left\{1+2\alpha\beta A+\frac{1+2\alpha\beta E}{1-\beta^2}y^2\right\}. \quad (3.5)$$

In the limit $\beta\rightarrow 1$, $y\rightarrow 0$, the cross section vanishes.

[7] S. D. Drell and R. H. Pratt, Phys. Rev. **125**, 1394 (1964).

[8] J. W. Motz (private communication); J. W. Motz, H. Olsen, and H. W. Koch (to be published); also L. A. Page, Rev. Mod. Phys. **31**, 759 (1959).

TABLE II. Comparison of S obtained in small-angle approximation (S_A) and (2.4) and by exact numerical calculation (S).

Z	13		79		80	
β	0.4		0.6889		0.4	
θ	S_A	S	S_A	S	S_A	S
5°	1.101×10^{-5}	1.108×10^{-5}	1.648×10^{-4}	1.628×10^{-4}	1.202×10^{-4}	1.250×10^{-4}
10°	-2.850×10^{-5}	-2.770×10^{-5}	1.198×10^{-3}	1.272×10^{-3}	-1.450×10^{-4}	-1.316×10^{-4}
15°	-1.964×10^{-4}	-1.931×10^{-4}	3.740×10^{-3}	3.949×10^{-3}	-6.833×10^{-4}	-7.540×10^{-4}

The asymmetry function is

$$S\simeq-\frac{2y}{(1-\beta^2)^{1/2}}\times\frac{\alpha\beta(B-H)}{\{1+2\alpha\beta A+[(1+2\alpha\beta E)/(1-\beta^2)]y^2\}}\equiv\frac{-[\alpha\beta y/(1-\beta^2)^{1/2}]P(\alpha,\beta)}{1+Q(\alpha,\beta)[y^2/(1-\beta^2)]},\quad(3.6)$$

where $P(\alpha,\beta)$ and $Q(\alpha,\beta)$ are defined by means of (3.6). In the limit $y\to0$, S also vanishes. However, for a given β it is not hard to see that there is an angle $y_{\max}$ at which S is a maximum, with the values

$$S_{\max}=\frac{-\alpha\beta P(\alpha,\beta)}{2[Q(\alpha,\beta)]^{1/2}},\quad y_{\max}=\left(\frac{1-\beta^2}{Q(\alpha,\beta)}\right)^{1/2}.\quad(3.7)$$

It is clear from (3.7) that the angle at which S is maximum approaches 180° as $\beta\to1$.

The above analysis is based on the approximation to order α^2. We may generalize (3.6) to all orders of α^2 and consider $P(\alpha,\beta)$ and $Q(\alpha,\beta)$ to include terms of all powers of α^2. The generalized $P(\alpha,\beta)$ and $Q(\alpha,\beta)$ can then be determined numerically by plotting $-\alpha\beta y/S(1-\beta^2)^{1/2}$ versus $y^2/(1-\beta^2)$. Such a plot is shown in Fig. 1 and the straight-line fit demonstrates the validity of (3.6) and shows that the values of $P(\alpha,\beta)$ and $Q(\alpha,\beta)$ are rather insensitive function of α and β.

It is possible that the striking behavior of the asymmetry function in the backward direction can be used as an experimental tool. The present calculation has been carried through under the assumption of a point Coulomb potential. It is well known that in scattering from actual nuclei, the effects of magnetic structure are dominant at backward angles. Deviation in the observations from those predicted by (3.6) may perhaps be used as a sensitive tool to explore this magnetic structure. These possibilities are being investigated.

IV. SMALL-ANGLE APPROXIMATION FOR A SCREENED FIELD

a. Analytical Behavior

In Sec. 3 of Paper I, we studied the behavior of the Coulomb amplitudes f and g (or F and G) for small θ. In actual scattering, however, the small-angle behavior is strongly modified by the effects of atomic screening. Exact calculations for a screened potential can only be done numerically by solving for the radial wave function in order to obtain the phase shifts and then summing the partial-wave series.[2,3]

A simple analytical approximation is suggested by the Born approximation[9] for a potential of the form

$$V(r)=(Ze^2/r)e^{-r/a}.\quad(4.1)$$

This modifies the cross section by the factor

$$\frac{(d\sigma/d\Omega)_{\text{screened}}}{(d\sigma/d\Omega)_{\text{Coulomb}}}=\frac{x^4}{(x^2+\frac{1}{2}\mu^2)^2},\quad(4.2)$$

with

$$\mu=1/ka,\quad(4.3)$$

but yields no information about the asymmetry. Comparison with actual calculations for relativistic scattering indicates that (4.3) is a poor representation of the effect of screening. For this reason we have extended the small angle calculation of Sec. 3 of Paper I to include the effect of screening in an approximate way.

Our starting point is the partial-wave expansions in (I-2.1). We shall introduce the effect of screening by imagining that the phase shifts up to a certain value of n (or $-n-1$) are the Coulomb phases properly modified by adding $q\ln ka=-q\ln\mu$, and vanish beyond this

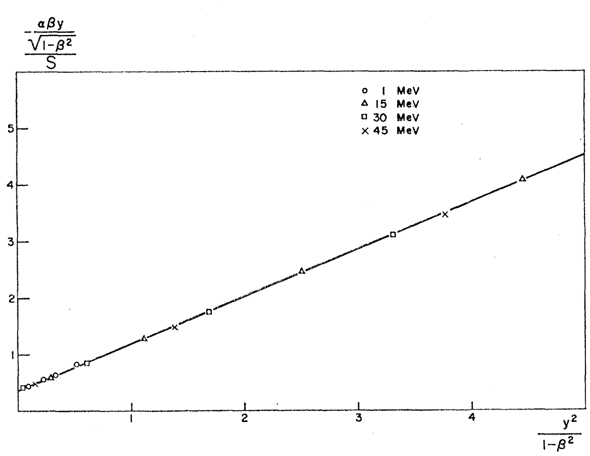

FIG. 1. The plot of $-\alpha\beta y/S(1-\beta^2)^{1/2}$ versus $y^2/(1-\beta^2)$ to determine $P(\alpha,\beta)$ and $Q(\alpha,\beta)$ in (3.6).

[9] G. Molière, Z. Naturforsch, **2a**, 133 (1947), R. H. Dalitz, Proc. Roy. Soc. (London) **A206**, 509 (1951).

value. In order to obtain analytic results however we shall modify this prescription by introducing the factor $e^{-\mu n}$ into the Coulomb expressions for $e^{2i\eta_n}-1$ and $e^{2i\eta_{-n-1}}-1$ to simulate this cutoff on the phase shifts η_n and η_{-n-1}. The parameter μ is given by (4.3) as can readily be verified from a WKB approximation for the phase shifts for large n. In our phenomenological considerations, μ will be taken as being small compared to 1.

The above considerations lead directly to the following expressions for the scattering amplitudes:

$$kfe^{-\mu/2}=\frac{i}{2}\Big[\sum_{n=1}^{\infty} n^2C_ne^{-\mu n}(P_ne^{\mu/2}+P_{n-1}e^{-\mu/2}) -iq'\sum_{n=1}^{\infty} nC_ne^{-\mu n}(P_ne^{\mu/2}-P_{n-1}e^{-\mu/2}) +\mu^{2iq}\sum_{n=0}^{\infty}(2n+1)e^{-\mu n}P_n\Big], \tag{4.4}$$

$$kge^{-\mu/2}=-\frac{i}{2}\frac{d}{d\theta}\Big[\sum_{n=1}^{\infty} nC_ne^{-\mu n}(P_ne^{-\mu/2}-P_{n-1}e^{\mu/2}) +iq'\sum_{n=1}^{\infty} C_ne^{-\mu n}(P_ne^{-\mu/2}+P_{n-1}e^{\mu/2})\Big]. \tag{4.5}$$

For small μ one obtains the approximate results:

$$kf\simeq J_2^+ +\frac{\mu}{2}J_2^- -iq'J_1^- -iq'\frac{\mu}{2}J_1^+ +\frac{i\mu^{2iq+1}}{(\mu^2+4x^2)^{3/2}}, \tag{4.6}$$

$$kg\simeq -xJ_2^+ -\frac{\mu}{2x}J_2^- +\frac{iq'}{x}J_1^- +\frac{iq'\mu x}{2}J_1^+, \tag{4.7}$$

where

$$q'=q/\gamma \quad\text{and}\quad \gamma=1/(1-\beta^2)^{1/2}, \tag{4.8}$$

and where

$$J_m^{\pm}=\frac{i}{2}\sum_{n=1}^{\infty} n^m(P_n\pm P_{n-1})C_ne^{-\mu n}. \tag{4.9}$$

The sums (4.9) are related to one another by the expressions

$$J_{m+1}^{\pm}=\pm(1\pm z)(dJ_m^{\mp}/dz) \tag{4.10}$$

which are analogous to (I-2.14).

The sums in (4.9) can be performed in the same way as those for $\mu=0$ in Sec. 3 of Paper I. One obtains directly

$$J_0^{\pm}=\frac{-i}{2\Gamma(1+2iq)}\int_0^1 dt(1-t)^{2iq}t^{-iq-1} \times\frac{(e^{\mu}\pm t)}{(e^{2\mu}-2zte^{\mu}+t^2)^{1/2}}, \tag{4.11}$$

where the term independent of z has been dropped since (4.6) and (4.7) involve at least one differentiation of (4.11) with respect to z. The integral in (4.11) can be converted as before to an integral from $-\infty$ to 0, and one over the contour C shown in Fig. 1 of Paper I, but the branch points at $t=e^{\pm i\theta}$ now move to $t=e^{\mu\pm i\theta}$.

The dominant behavior for small μ and θ is contained in the contour integral. This integral cannot be evaluated analytically for arbitrary μ and θ; we shall therefore evaluate it in the two cases (i) $\mu\ll\theta\ll1$ and (ii) $\theta\ll\mu\ll1$.

Case (i) $\mu\ll\theta\ll1$

Setting $t=1+2vx$, and expanding for small μ/x, one finds eventually:

$$J_0^+\simeq\frac{1}{2q}x^{2iq}e^{-2i\sigma_0}\Big[1+iq\frac{\mu}{x}e^{i\psi}+\mathcal{O}\Big(\frac{\mu^2}{x^2}\Big)\Big] \tag{4.12}$$

and

$$J_0^-\simeq\frac{ix^{2iq+1}}{2iq+1}e^{i\psi-2i\sigma_0}\Big[1+\mathcal{O}\Big(\frac{\mu}{x}\Big)\Big]. \tag{4.13}$$

Using (4.10), one is ultimately led to the amplitudes

$$kf\simeq-\frac{iqF_0}{x^2}\Big[1+\frac{\mu}{x}\frac{(iq-\frac{1}{2})^2}{iq}e^{i\psi}+\frac{i\mu}{4qx}e^{i\chi}+\mathcal{O}\Big(\frac{\mu^2}{x^2}\Big)\Big], \tag{4.14}$$

$$kg\simeq-iq\Big(1-\frac{1}{\gamma}\Big)\frac{F_0}{x}\Big[1+\frac{\mu}{x}(iq-\tfrac{1}{2})e^{i\psi}+\mathcal{O}\Big(\frac{\mu^2}{x^2}\Big)\Big], \tag{4.15}$$

where $\chi=2\sigma_0-2q\ln(x/\mu)$. This leads to

$$\frac{(d\sigma/d\Omega)_{\text{screened}}}{(d\sigma/d\Omega)_{\text{Coulomb}}} \simeq1+\frac{2\mu}{x}\operatorname{Re}\Big[e^{i\psi}\frac{(iq-\frac{1}{2})^2}{iq}+\frac{1}{4q}e^{i\chi}\Big]+\mathcal{O}\Big(\frac{\mu^2}{x^2}\Big), \tag{4.16}$$

$$\simeq1-\frac{2\mu}{x}\Big[\cos\psi-\sin\psi\Big(\frac{1}{4q}-q\Big)+\frac{1}{4q}\sin\chi\Big]+\mathcal{O}\Big(\frac{\mu^2}{x^2}\Big),$$

and

$$S\simeq\mu\Big(\sin\psi+\frac{\cos\psi}{2q}-\frac{\cos\chi}{2q}\Big)\Big(1-\frac{1}{\gamma}\Big)\Big[1+\mathcal{O}\Big(\frac{\mu}{x}\Big)\Big]. \tag{4.17}$$

The following points should be noted from (4.16) and (4.17) which are valid in the region $\mu\ll x\ll1$:

(1) The correction to the cross section due to screening is of order μ/x rather than μ^2/x^2 as suggested by the Born approximation result in (4.2). Moreover, it contains an oscillatory term.

(2) The asymmetry is independent of angle in this region, except for the oscillatory term.

(3) The asymmetry vanishes in the nonrelativistic limit $\beta\to0$, as expected.

Case (ii) $\theta\ll\mu\ll1$

In this case it is appropriate to obtain the integral representations for $J_{1,2}^{\pm}$ corresponding to (4.11) and to proceed to the limit $z\to1$. In this way one is led to

$$J_1^+ \simeq \tfrac{1}{2} i\Gamma(1-2iq)\mu^{2iq-1}[1+\mathcal{O}(x^2/\mu^2)]\,, \qquad (4.18)$$

$$J_1^-/2x^2 \simeq -\tfrac{1}{2} i\Gamma(2-2iq)\mu^{2iq-2}[1+\mathcal{O}(x^2/\mu^2)]\,, \qquad (4.19)$$

for the leading terms for small μ as $x\to 0$, with corresponding expressions for $J_2^\pm$ obtained from (4.10). From (4.6) and (4.7), one then finds

$$kf \simeq i\mu^{2iq-2}[\Gamma(2-2iq)+1][1+\mathcal{O}(x^2/\mu^2)]\,, \qquad (4.20)$$

$$kg \simeq -xq\left(1-\frac{1}{\gamma}\right)\mu^{2iq-2}\Gamma(2-2iq)\left[1+\mathcal{O}\left(\frac{x^2}{\mu^2}\right)\right], \qquad (4.21)$$

which leads to

$$\frac{(d\sigma/d\Omega)_{\text{screened}}}{(d\sigma/d\Omega)_{\text{Coulomb}}} = \frac{4x^2}{q^2}|1+\Gamma(2-2iq)|^2\left[1+\mathcal{O}\left(\frac{x^2}{\mu^2}\right)\right] \qquad (4.22)$$

and

$$S = 2xq\left(1-\frac{1}{\gamma}\right)\mathrm{Re}\left[\frac{\Gamma(2-2iq)}{1+\Gamma(2-2iq)}\right]\left[1+\mathcal{O}\left(\frac{x^2}{\mu^2}\right)\right]. \qquad (4.23)$$

The following points should be noted from (4.22) and (4.23), which are valid in the region $x \ll \mu \ll 1$:

(1) The correction to (4.22) for $x=0$ is of order x^2/μ^2, similar to the dependence contained in the Born result in (4.2).

(2) The asymmetry varies linearly with angle near $x=0$ in the screened case as opposed to the behavior proportional to x^2 in the Coulomb case.

(3) The asymmetry vanishes in the nonrelativistic limit $\beta\to 0$, as expected.

(4) The orders of magnitude of (4.17) and (4.23) are the same for $x\sim\mu$.

The above approximations for the screened amplitudes have been compared with actual numerical computations and the agreement has been reasonable, considering the nature of the approximations. As an example, we list in Table III

$$x\left[\frac{|g_{\text{Coulomb}}|}{|g_{\text{screened}}|}-1\right]$$

for several values of θ, with $Z=79$, $\beta=0.688$, using the modified Hartree potential.[3] According to (4.15) the result should be independent of x, which it appears to be. Moreover, the value of μ suggested by this procedure is

$$\mu = 0.025 \text{ rad} \simeq 1.4°\,,$$

in approximate agreement with (4.3) with a value for a consistent with the range of the Hartree potential.

Lest one get the impression that the procedure is without any inconsistencies, let us note the following points:

TABLE III. Comparison of $|g_{\text{Coulomb}}/g_{\text{screened}}|$ with the small-angle approximation to determine the equivalent screening parameter μ.

θ	$2kg_{\text{Coulomb}}$	$2kg_{\text{screened}}$	$x\left[\frac{\lvert g_{\text{Coulomb}}\rvert}{\lvert g_{\text{screened}}\rvert}-1\right]$
6°	$-1.690+i4.230$	$-1.174+i2.849$	0.0250
10°	$-2.697+i0.804$	$-1.946+i0.947$	0.0262
14°	$-1.983-i0.667$	$-1.735-i0.157$	0.0245
18°	$-1.205-i1.224$	$-1.289-i0.743$	0.0242
22°	$-0.586-i1.381$	$-0.853-i1.032$	0.0230
26°	$-0.127-i1.357$	$-0.480-i1.146$	0.0217
30°	$0.207-i1.253$	$-0.175-i1.165$	0.0204

(1) A similar behavior for $d\sigma/d\Omega$, as given in (4.16), also is confirmed, but with a slightly different value of μ. This comparison is made more difficult by the presence of the $\ln(\mu/x)$ term.

(2) In performing the comparisons one must take ratios of (4.14) and (4.15) with the relativistic unscreened results. If this is not done, errors to the bracket of order x occur. Since one must use relatively large values of x (θ up to 30°) in order to minimize the effect of μ^2/x^2 terms, this correction of order x must be included by using the unscreened results.

(3) The phases of f_{screened} and g_{screened}, may also be compared with f_{Coulomb} and g_{Coulomb} to obtain the equivalent values for μ. This leads to somewhat different values for μ than that obtained from Table III. But this is not too surprising since these phases are determined by the comparison of the screened and Coulomb phase shifts for low n, whereas the value of μ in the brackets of (4.14) and (4.15) is determined by the rate at which the phase shifts tend to zero for high n.

(4) The value of S predicted by (4.17) must also be compared with S_{Coulomb} for the reasons given in (2) above. Even with this modification, however, (4.17) does not represent more than an order of magnitude estimate of S. The reason is that S is sensitive to the *relative* phase of f and g, and the different values of μ which are appropriate to each term modify the result in (4.17).

(5) The above discussion should also apply to the reliability of the forms for $x\ll\mu$ given in (4.20)–(4.23).

In spite of the cautions listed above, we have succeeded in obtaining the approximate analytic behavior at small angles of the cross section and asymmetry for a screened potential. These should serve as a useful guide to the comparison of numerical calculations with those for the unscreened case.

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